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A convex body with chaotic random convex hull

Olivier Devillers*, Marc Glisse†, Rémy Thomasse‡

Project-Team Geometrica

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Abstract: The asymptotic behavior of the size of the convex hull of uniformly random points in a convex body in $\mathbb{R}^d$ is known for polytopes and smooth convex bodies. These are the lower and the upper bound for a general convex body. In this paper, we exhibit an example of convex body whose size of the random convex hull alternates behavior close to the lower and to the upper bound for some values of the number of points arbitrary big.

Key-words: Convex hull, point distribution, random analysis

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* Projet Geometrica, INRIA Sophia Antipolis - Méditerranée
† Projet Geometrica, INRIA Saclay - Île de France
‡ Projet Geometrica, INRIA Sophia Antipolis - Méditerranée

RESEARCH CENTRE
SOPHIA ANTIPOLIS – MÉDITERRANÉE

2004 route des Lucioles - BP 93
06902 Sophia Antipolis Cedex

Un ensemble de points aléatoires dont l'enveloppe convexe a un comportement chaotique

Résumé : Le comportement asymptotique de la taille de l'enveloppe convexe d'un ensemble de points uniformément distribué dans un convexe de $\mathbb{R}^d$ est connu dans le cas des polytopes et des convexes lisses. Ces deux cas étant les bornes inférieures et supérieures entre lesquels peut varier la taille de l'enveloppe convexe dans le cas d'un convexe quelconque. Dans cet article, nous construisons un convexe telle que la taille de l'enveloppe d'un ensemble de point aléatoire a un comportement chaotique qui oscille entre un comportement proche de la borne inférieure et proche de la borne supérieure quand le nombre de points tend vers l'infini.

Mots-clés : Enveloppe convexe, distribution de points, analyse aléatoire

1 Introduction

Consider a sequence of points in a convex body in dimension d whose convex hull is dynamically maintained when the points are inserted one by one, the convex hull size may increase, decrease, or remain constant when a new point is added. Studying the expected size of the convex hull when the points are evenly distributed in the convex is a classical problem of probabilistic geometry that yields some surprising facts. For example, although it seems quite natural to think that the expected size of the convex hull is increasing with n the number of points, this fact is only formally proven for n big enough [4]. The asymptotic behavior of the expected size is known to be polylogarithmic for a polytopal body and polynomial for a smooth one. If for a polytope or a smooth body, the asymptotic behavior is *somehow* "nice", for "most" convex bodies the behavior is unpredictable [1, corollary 3]. It is possible to construct strange convex objects that have no such nice behaviors and this note exhibits a convex body, such that the behavior of the expected size of a random polytope oscillates between the polytopal and smooth behaviors when n increases.

More formally, let K be a convex body in $\mathbb{R}^d$ and $(x_1, \dots, x_n)$ a sample of n points chosen uniformly and independently at random in K. Let K_n be the convex hull of these points and $f_0(K_n)$ the number of vertices of K_n.
It is well known [2, 5] that if P is a polytope, then

$$\mathbb{E}f_0(P_n) = c_{d,P} \log^{d-1} n + o(\log^{d-1} n) \tag{1}$$

and if K is a smooth convex body (i.e with $\mathcal{C}^2$ boundary with a positive Gaussian curvature), then

$$\mathbb{E}f_0(K_n) = c_{d,K} n^{\frac{d-1}{d+1}} + o(n^{\frac{d-1}{d+1}}) \tag{2}$$

where $c_{d,P}$ and $c_{d,K}$ are constants depending only on d and on the convex body.
These are the two extreme behaviors : every random polytope of a convex body in $\mathbb{R}^d$ has a behavior between (1) and (2) for n large enough [1, corollary 3].
For general convex bodies, we cannot expect such a beautiful formula.

Theorem 1 (Bàràny-Larman, 1988)**.** *For any function $G(n) \to_{n\to\infty} \infty$ and for most (in the Baire category sense) convex bodies K in $\mathbb{R}^d$,*

$$G(n) \log^{d-1} n > \mathbb{E}f_0(K_n) \tag{3}$$

for infinitely many n and

$$G(n)^{-1} n^{\frac{d-1}{d+1}} < \mathbb{E}f_0(K_n) \tag{4}$$

for infinitely many n.

Note that this *"most"* does not contain convex polytopes and smooth convex bodies, which are the most used in practice.

In this paper, we present an explicit example of a convex body which has this chaotic behavior.

Notations Let's introduce some notations used in this paper:

- $\mathcal{V}(K)$ will denote the $d-$dimensional volume of the convex body K;
- $K \oplus L$ will denote the Minkowski sum of K and L, defined as

$$J \oplus K = \{x + y \mid x \in J, y \in K\}; \tag{5}$$

- $d_H(J,K)$ will denote the Hausdorff distance of the convex bodies K and L, defined as
$$d_H(J,K) = \min\{r \in \mathbb{R}^+ \,| J \subset K \oplus B_r, K \subset J \oplus B_r\} \tag{6}$$
where B_r is the Euclidean ball centered in 0 with radius r in $\mathbb{R}^d$.

2 Approximations of convex bodies

In this section, we present an intermediate lemma about random polytopes of close (in terms of Hausdorff distance) convex bodies. Let K be a convex body in $\mathbb{R}^d$ and K_n be a random polytope in K. We want to show that if L is an approximation of K with small Hausdorff distance, L_n is approximating the asymptotic behavior of K_n for some value of n.
Let's assume that the expected size of K_n is in $c_{d,K}g(n,d) + o(g(n,d))$, where $c_{d,K}$ is a constant and g some function.
Then, for every close-enough compact set L containing K, L_n has an expected size as close as we want from $c_{d,K}g(n,d)$ for values of n as big as we want.
The idea of this lemma is very simple: if L is very close to K, the volume in $L \setminus K$ is small, and points chosen uniformly in L are very unlikely to be in $L \setminus K$. Then, even if asymptotically the expected size of L_n is different from the size of K_n, there exists some n where the expected size of L_n is as close as we want to $c_{d,K}g(n,d)$.

Lemma 2. *Let K be a convex body in $\mathbb{R}^d$ such that*
$$\mathbb{E}f_0(K_n) = c_{d,K}g(n,d) + o(g(n,d)). \tag{7}$$
Let $\varepsilon > 0$ and $N \in \mathbb{N}$.
Then, there exist $p > N$ and $\alpha > 0$ such that for any compact set L containing K with $d_H(K,L) < \alpha$,
$$\frac{\mathbb{E}f_0(L_p)}{c_{d,K}g(p,d)} \in [1-\varepsilon, 1+\varepsilon]. \tag{8}$$

Proof. First, for all $n \in \mathbb{N}^*$
$$\mathbb{E}f_0(L_n) = \mathbb{P}(L_n \subset K)\mathbb{E}(f_0(L_n)|L_n \subset K) + \mathbb{P}(L_n \not\subset K)\mathbb{E}(f_0(L_N)|L_n \not\subset K). \tag{9}$$
As the points are uniformly distributed, $\mathbb{E}(f_0(L_n)|L_n \subset K) = \mathbb{E}(K_n)$.
Using (7), let's choose p such that
$$\frac{\mathbb{E}f_0(K_p)}{c_{d,K}g(p,d)} \in \left[1-\frac{\varepsilon}{2}, 1+\frac{\varepsilon}{2}\right], \tag{10}$$
then
$$\frac{\mathbb{P}(L_p \subset K)\mathbb{E}(f_0(L_p)|L_p \subset K)}{c_{d,K}g(p,d)} \leq 1+\frac{\varepsilon}{2}. \tag{11}$$
As
$$\mathbb{P}(L_p \not\subset K) = 1 - \mathbb{P}(L_p \subset K) = 1 - \left(\frac{\mathcal{V}(K)}{\mathcal{V}(L)}\right)^p \tag{12}$$
we get
$$\mathbb{P}(L_p \not\subset K)\mathbb{E}(f_0(L_p)|L_p \not\subset K) \leq \left(1 - \left(\frac{\mathcal{V}(K)}{\mathcal{V}(L)}\right)^p\right)p. \tag{13}$$

Now, as $1 \geq \frac{\mathcal{V}(K)}{\mathcal{V}(L)} \geq \frac{\mathcal{V}(K)}{\mathcal{V}(K \oplus B_\alpha)} \to_{\alpha \to 0} 1$ we can choose α such that

$$\left(\frac{\mathcal{V}(K)}{\mathcal{V}(L)}\right)^p \geq \max\left(1 - c_{d,K}\frac{g(p,d)}{p}\frac{\varepsilon}{2}, 1 - \frac{\varepsilon}{2}\right). \tag{14}$$

Finally,

$$\mathbb{E}(f_0(L_p)) \leq c_{d,K}g(p,d)\left(1 + \frac{\varepsilon}{2}\right) + c_{d,K}g(p,d)\frac{\varepsilon}{2} = c_{d,K}g(p,d)(1+\varepsilon). \tag{15}$$

For the lower bound, using (10) and (14) we get

$$\begin{aligned}
\mathbb{E}f_0(L_p) &\geq \mathbb{P}(L_p \subset K)\mathbb{E}(f_0(L_n)|L_n \subset K) \\
&\geq c_{d,K}g(p,d)\left[\left(\frac{\mathcal{V}(K)}{\mathcal{V}(L)}\right)^p (1 - \frac{\varepsilon}{2})\right] \\
&\geq c_{d,K}g(p,d)\left[\left(\frac{\mathcal{V}(K)}{\mathcal{V}(L)}\right)^p - \frac{\varepsilon}{2}\right] \\
&\geq c_{d,K}g(p,d)\left(1 - \frac{\varepsilon}{2} - \frac{\varepsilon}{2}\right) \\
&= c_{d,K}g(p,d)(1-\varepsilon). \qquad (16)
\end{aligned}$$

Inequalities (16) and (15) prove the lemma. □

3 Construction of the convex body

Given an increasing function G, we want to construct a convex body in $\mathbb{R}^d$ where the size of a convex hull of random points has a chaotic behavior between $\log^{d-1} n$ and $n^{\frac{d-1}{d+1}}$ on some values arbitrarily big. More formally,

Theorem 3. *Let $G : \mathbb{N}^* \to \mathbb{R}_+^*$ an increasing function such that $G(n) \to_{n\to\infty} \infty$. We can construct a convex body K such that:*
For all $N \in \mathbb{N}^*$, there exist $M_1, M_2 > N$, where

$$\mathbb{E}f_0(K_{M_1}) < G(M_1)\log^{d-1} M_1 \tag{17}$$

and

$$\mathbb{E}f_0(K_{M_2}) > G(M_2)^{-1}\ M_2^{\frac{d-1}{d+1}}. \tag{18}$$

Proof. The main idea of the proof is, starting from a convex body $K^{(0)}$, to iterate smooth and polytopal approximations. Lemma 2 will give us some number of points where the behavior of the random convex body will be very close to $n^{\frac{d-1}{d+1}}$ (which is the behavior for smooth convex bodies) or very close to $\log^{d-1} n$ (witch is the behavior for polytopes).

Iterations We create an increasing sequence of convex bodies starting from the unit ball, made of polytopal and smooth approximations.
Let's define K^0 as the unit ball.
For all $n \in \mathbb{N}^*$, $K^{(n)}$ is an approximation of $K^{(n-1)}$ where $d_H(K^{(n-1)}, K^{(n)}) < \beta_{n-1}$, with $(\beta_i)_{i\in\mathbb{N}}$ some decreasing sequence, as shown in Figure 1.

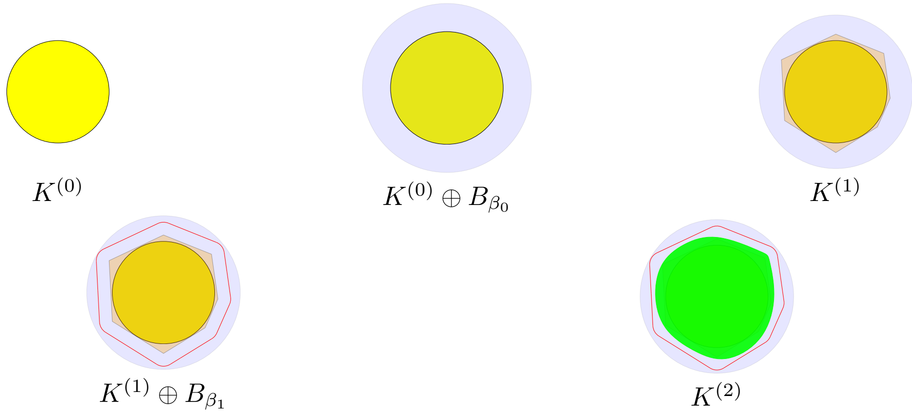

Figure 1: Iterations are made of polygonal and smooth approximations

- If n is odd, $K^{(n-1)}$ is a smooth convex body, so $K^{(n)}$ is a convex polytope. Let's choose $q_n > n$, such that

$$\frac{\mathbb{E}f_0(K^{(n-1)})}{q_n^{\frac{d-1}{d+1}}} > \frac{2}{G(q_n)}. \tag{19}$$

We define

$$\varepsilon_n := 1 - \frac{2}{c_{d,K^{(n-1)}}G(q_n)}. \tag{20}$$

Using Lemma 2,with $\varepsilon = \varepsilon_n$, there exist α_{n-1} and $p_n > q_n$ such that for every compact set L containing $K^{(n-1)}$ with $d_H(K^{(n-1)}, L) < \alpha_{n-1}$,

$$\mathbb{E}f_0(L_{p_n}) > c_{d,K^{(n-1)}} p_n^{\frac{d-1}{d+1}}(1-\varepsilon_n). \tag{21}$$

Therefore,

$$\begin{aligned}\mathbb{E}f_0(L_{p_n}) &> c_{d,K^{(n-1)}} p_n^{\frac{d-1}{d+1}}(1-\varepsilon_n)\\ &> p_n^{\frac{d-1}{d+1}} G(q_n)^{-1}\\ &> p_n^{\frac{d-1}{d+1}} G(p_n)^{-1}.\end{aligned} \tag{22}$$

Now let's define $\beta_{n-1} := \min(\frac{\alpha_{n-1}}{2}, \frac{\beta_{n-2}}{2})$ if $n > 1$ and $\beta_0 := \frac{\alpha_0}{2}$. We define $K^{(n)}$ as a convex polytope with $d_H(K^{(n-1)}, K^{(n)}) < \beta_{n-1}$, so (22) works for $L = K^{(n)}$.

- If n is even, $K^{(n-1)}$ is a convex polytope, so $K^{(n)}$ is a smooth approximation.

Let's choose q_n such that

$$\frac{\mathbb{E}f_0(K^{(n-1)})}{\log^{d-1} q_n} < \frac{G(q_n)}{2} \tag{23}$$

and define

$$\varepsilon_n := \frac{G(q_n)}{2c_{d,K^{(n-1)}}} - 1. \tag{24}$$

Using Lemma 2 with $\varepsilon = \varepsilon_n$, there exist α_{n-1} and $p_n > q_n$ such that for every compact set with $d_H(K^{(n-1)}, L) < \alpha_{n-1}$,

$$\mathbb{E}f_0(L_{p_n}) < c_{d,K^{(n-1)}} \log^{d-1}(p_n)(1+\varepsilon_n). \tag{25}$$

Finally,

$$\begin{aligned}\mathbb{E}f_0(L_{p_n}) &< G(q_n)\log^{d-1}(p_n)\\ &< G(p_n)\log^{d-1}(p_n).\end{aligned} \tag{26}$$

Again, we define $\beta_{n-1} = \min(\frac{\alpha_{n-1}}{2}, \frac{\beta_{n-2}}{2})$. We define $K^{(n)}$ as a smooth approximation of $K^{(n-1)}$ such that $d_H(K^{(n-1)}, K^{(n)}) < \ \beta_{n-1}$, so (26) works for $L = K^{(n)}$.

Note that for all $m > n \in \mathbb{N}$,

$$d_H(K^{(n)}, K^{(m)}) \leq \sum_{k=n}^{m-1} d_H(K^{(k)}, K^{(k+1)}) < \sum_{k=n}^{m-1} \beta_k \tag{27}$$

$$\leq \sum_{k=0}^{m-n-1} \frac{\beta_n}{2^k} \leq 2\beta_n \leq \alpha_n. \tag{28}$$

That means for all $m > n$, the property (22) or (26) (depending on the evenness of n) are also true for $K^{(m)}$.
Now, defining $K = \overline{\cup_{i=0}^{\infty} K^{(i)}}$, the property (22) and (26) are true for arbitrary $n \in \mathbb{N}$ with $L = K$, by considering $K^{(n)}$ and $K^{(n+1)}$.
As we can choose q_n as big as we want for any n (it will just decrease α_{n-1}), we can choose this sequence to be increasing. As a result, $\mathbb{E}f_0(K_n)$ will have a chaotic behavior within $n^{\frac{d-1}{d+1}}/G(n)$ and $G(n)\log^{d-1} n$, as shown in Figure 2.

□

Concluding remarks We have constructed a convex body K such that the expected size of the convex hull of a random polytope in K has a chaotic behavior. This construction is the limit of a sequence of bodies $(K^{(i)})$ that alternate polytopes and smooth shapes so it is difficult to provide an explicit description of K, in this note we just show that constructing such a sequence is possible by a repeated application of Lemma 2 but there is no obstacle, except long and painful computations, to a more constructive version with explicit description of the sequence. Notice that in such a case the complexity of $K^{(i)}$ will be increasing quite rapidly. Actually, since $K^{(i)}$ is constrained in a slab of width β_i around $K^{(i-1)}$, the size of $K^{(i)}$ can be lower bounded for polytopes, see [3]: $|K^{(i)}| = \Omega\left(\beta_i^{-\frac{d-1}{2}}\right)$ and since $\beta_i < \frac{\alpha_0}{2^i}$ we get, at least, an exponential behavior for the size of $K^{(i)}$. Even with a constructive description of the $K^{(i)}$, the description of K as the limit of the $K^{(i)}$ will remain quite abstract, but will allow to develop a membership test, given a point p, $p \in \text{int}(K)$ can be decided by computing the sequence $K^{(i)}$ up to an index where $p \in K^{(i)}$ or $p \notin K^{(i)} \oplus B_{\beta_i}$.

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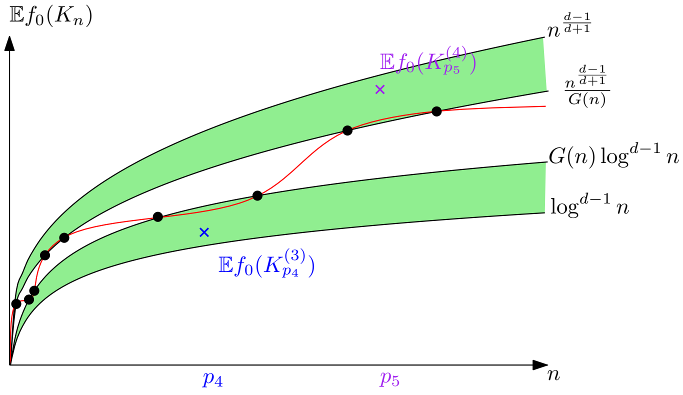

Figure 2: The expected size of the random polytope of K.

References

[1] Imre Bárány. Intrinsic volumes and f-vectors of random polytopes. *Mathematische Annalen*, 285(4):671–699, 1989.

[2] Imre Bárány. Random points and lattice points in convex bodies. *Bull. Amer. Math. Soc. (N.S.)*, 45(3):339–365, 2008. doi:10.1090/S0273-0979-08-01210-X.

[3] Károly Böröczky Jr. Polytopal approximation bounding the number of k-faces. *Journal of Approximation Theory*, 102(2):263 – 285, 2000. doi:10.1006/jath.1999.3413.

[4] Olivier Devillers, Marc Glisse, Xavier Goaoc, Guillaume Moroz, and Matthias Reitzner. The monotonicity of f-vectors of random polytopes. *Electron. Commun. Probab.*, 18:no. 23, 1–8, 2013. doi:10.1214/ECP.v18-2469.

[5] Matthias Reitzner. Random polytopes. In *New perspectives in stochastic geometry*, 45–76. Oxford Univ. Press, Oxford, 2010.

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